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Solution of Two-Point Boundary-Value Problems Using Lagrange Implicit Function Theorem

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Introduction

TWO-POINT boundary-value problems (TPBVPs) form an important ingredient in the solution of several multiphysics modeling and control analyses, including but not limited to guidance, navigation, and control problems of aerospace engineering. Several techniques to solve TPBVPs have been developed so far and the workhorse at the core of each has been Newton's method (cf. Bryson and Ho [1]). In this Note, we present an implicit derivative Newton's shooting method to solve a class of two-point boundary-value problems, most often encountered in the solution of optimal control problems.

To motivate the developments of the paper, consider the optimal control problem

$$\min_{\mathbf{u}(t)} J = \varphi(\mathbf{x}_0, \mathbf{x}_f, t_0, t_f) + \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u}, t)\mathrm{d}t \tag{1}$$

subject to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2}$$

with $\mathbf{x}(t) \in \mathbb{R}^n \times \mathbb{R}^+$, $\mathbf{u}(t) \in \mathbb{R}^r \times \mathbb{R}$, and $x_1(t_0), \ldots, x_{q_0}(t_0)$ specified at the known initial time t_0 [i.e., $x_{q_0+1}(t_0), \ldots, x_n(t_0)$ components of the initial state vector unspecified] and given that certain components of the state vector at the terminal time lie on the manifold:

$$\boldsymbol{\psi}_f(t_f, x_1(t_f), \ldots, x_{q_f}(t_f), \mathbf{p}_f) = 0 \tag{3}$$

where $\boldsymbol{\psi}_f \in R^{q_f}$. Defining the Hamiltonian $H := L + \boldsymbol{\lambda}^T\mathbf{f}$, the necessary conditions for extrema using the standard variational approach to functional minimization problems [1] are given by

$$H_{\mathbf{u}} = 0, \qquad \dot{\mathbf{x}}(t) = H_{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}}(t) = -H_{\mathbf{x}} \tag{4}$$

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together with the transversality conditions

$$\frac{\partial \Phi}{\partial x_{j_1}(t_0)} + \lambda_{j_1}(t_0) = 0 \tag{5}$$

$$\frac{\partial \Phi}{\partial x_{j_2}(t_f)} - \lambda_{j_2}(t_f) = 0 \tag{6}$$

where

$$j_1 = q_0 + 1, \ldots, n, \qquad j_2 = q_f + 1, \ldots, n,$$
$$\Phi(t_0, t_f, \mathbf{x}_0, \mathbf{x}_f, \boldsymbol{\nu}_f, \mathbf{p}_f) := \varphi + \boldsymbol{\nu}_f^T\boldsymbol{\psi}_f, \qquad \boldsymbol{\nu}_f \in R^{q_f}$$

are the Lagrange multipliers associated with the hard terminal constraints specified in the problem [i.e., Eqs. (3)]. Thus, the TPBVP is obtained naturally. A strategy to solve this problem guesses initial states (and/or) costates $(\mathbf{x}_0^G, \boldsymbol{\lambda}_0^G)$ that are consistent with the known initial state vector components, while satisfying the transversality condition at initial time (5). In other words, for the given optimal control problem, one would need to guess q_0 of the initial costates (which we denote by $\boldsymbol{\lambda}_0^G$) and $n - q_0$ of the initial states (denoted by $\mathbf{x}_0^G$: the components unspecified in the problem statement). These initial guesses are used to integrate the state/costate differential equations, and the conditions at terminal time (3) and (6) are checked for satisfaction to arbitrary tolerance. The initial guesses are updated if the convergence tolerances are not met. This procedure is known as the shooting method and is used frequently to solve TPBVPs. For the problems with general boundary conditions considered previously, the shooting procedure is less straightforward. This Note develops a systematic procedure to set up a shooting method in such situations, using the Lagrange implicit function theorem. The procedure is subsequently employed in the solution of an example orbit transfer problem.

Lagrange Implicit Function Theorem: Application to Solution of the TPBVP

The Lagrange implicit function theorem [2–7] is an important result in analysis facilitating several theoretical and practical applications in applied mathematics and engineering. The theorem is stated as follows:

Lagrange's Implicit Function Inversion Theorem: Given the equation $f(x) = y$, where f is analytic at $x = a$ with $df/dx \neq 0$, then the *inverse function* is $x = g(y)$, where g is analytic at $y = b = f(a)$, Lagrange found the following general result:

$$x = g(y) = a + \sum_{n=1}^{\infty} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{x-a}{f(x)-b}\right)^n \Bigg|_{\substack{x=a \\ y(a)=b}} \left(\frac{(y-b)^n}{n!}\right)$$

Convergence depends on $f(x)$.

Several versions of this classical theorem exist in the literature, and the theorem has foundationally enabled the existence of solutions to ordinary differential and algebraic (transcendental) equations (for theoretical considerations, cf. [3,7–9]). The power of the theorem lies entirely in its generality, and the applications spanning several spheres of engineering are a testimony to this fact. The theorem and its proof are intimately connected to the idea of *differentiation of implicit functions*. We now apply this theorem to set up Newton's iteration for the solution of the optimal control problem set up in the previous section.

Assume that the shooting method mentioned in the previous section is used to solve for the initial costate conditions as independent variables. We note in passing that this choice of independent variables solves the problem entirely for a local extremum, although some of the initial states are unspecified, owing to the transversality conditions (5). Then the conditions to be satisfied at the terminal time [i.e., Eqs. (3) and (6)] are implicit functions of the initial costates we are trying to determine. The n conditions at terminal time can thereby be arranged in a vector form given by

$$\boldsymbol{\Theta}\,(t_f, \mathbf{x}_f, \mathbf{p}_f, \boldsymbol{\lambda}(t_f)) := \begin{bmatrix} \boldsymbol{\psi}_f(t_f, x_1(t_f), \ldots, x_{q_f}(t_f), \mathbf{p}_f) \\ \frac{\partial \Phi}{\partial x_{q_f+1}(t_f)} - \lambda_{q_f+1}(t_f) \\ \vdots \\ \frac{\partial \Phi}{\partial x_n(t_f)} - \lambda_n(t_f) \end{bmatrix} = 0 \tag{7}$$

Therefore, it is a nonlinear functional inversion problem with n conditions [Eq. (7)] and n unknowns to determine. By making use of the transversality conditions at initial time t_0 [Eq. (5)], without loss of generality, the unknown state vector components can be traded in favor of the corresponding costates. That is to say that in the present problem, $\mathbf{x}_0^G$ and $\boldsymbol{\lambda}_0^G$ can be traded in favor of an n-dimensional vector of initial costates $\boldsymbol{\lambda}_0$. Writing the first-order Taylor series expansion of the implicit functions about a nominal initial costate $\boldsymbol{\lambda}_0^*$,

$$\begin{aligned} &\boldsymbol{\Theta}(t_f, \mathbf{x}_f(\boldsymbol{\lambda}_0^* + \delta\boldsymbol{\lambda}_0), \mathbf{p}_f, \boldsymbol{\lambda}_f(\boldsymbol{\lambda}_0^* + \delta\boldsymbol{\lambda}_0)) \\ &\quad = \boldsymbol{\Theta}(t_f, \mathbf{x}_f(\boldsymbol{\lambda}_0^*), \mathbf{p}_f, \boldsymbol{\lambda}_f(\boldsymbol{\lambda}_0^*)) + \left[\frac{d\boldsymbol{\Theta}}{d\boldsymbol{\lambda}_0^a}\right]_{\boldsymbol{\lambda}_0=\boldsymbol{\lambda}_0^*} \delta\boldsymbol{\lambda}_0^a + \text{HOT} \end{aligned} \tag{8}$$

where $\delta\boldsymbol{\lambda}_0^a$ denotes admissible variation of the components of the initial costate vector and HOT denotes the high-order terms neglected in the Taylor series expansion. The transversality conditions corresponding to the free initial states introduce a definitive subspace with dependence (linear to first order) between the variations of the states and the costates. This subspace guarantees the existence of solutions of the two-point boundary-value problem. Making the classical Newton iteration-based shooting method aware of this subspace was found to enhance the convergence of the algorithm. Correcting the initial costate vector from among the subspace of admissible variations of the initial costate function [from Eq. (8)] characterizes the method outlined in this Note. If the initial state conditions is fixed in the problem statement, we are naturally led to the algorithm presented in chapter 7 of Bryson and Ho [1], as all variations are admissible in this case. Further note that $\boldsymbol{\Theta} \in \mathbb{R}^n$ is independent of the terminal Lagrange multiplier $\boldsymbol{\nu}_f$. This is due to the functional dependence of the terminal constraint expression $\boldsymbol{\psi}_f(t_f, x_1(t_f), \ldots, x_{q_f}(t_f), \mathbf{p}_f)$ purely on $t_f, x_1(t_f), \ldots, x_{q_f}(t_f)$, and $\mathbf{p}_f$ [from the class of optimal control problems considered in Eq. (3)].

Consequently,

$$\frac{\partial \Phi}{\partial x_j(t_f)} - \lambda_j(t_f) = \frac{\partial \varphi}{\partial x_j(t_f)} - \lambda_j(t_f) \quad \forall\ j = q_f + 1, \ldots, n \tag{9}$$

The sensitivity (implicit) of the final conditions with respect to the initial costate vector is calculated using the implicit function theorem as

$$\frac{d\boldsymbol{\Theta}}{d\boldsymbol{\lambda}_0^a} = \frac{\partial \boldsymbol{\Theta}}{\partial \boldsymbol{\lambda}_0^a} + \frac{\partial \boldsymbol{\Theta}}{\partial \boldsymbol{\lambda}_f}\left(\frac{\partial \boldsymbol{\lambda}_f}{\partial \boldsymbol{\lambda}_0^a}\right) + \frac{\partial \boldsymbol{\Theta}}{\partial \mathbf{x}_f}\left(\frac{\partial \mathbf{x}_f}{\partial \boldsymbol{\lambda}_0^a}\right) \tag{10}$$

where the first-order perturbation matrices $\partial \mathbf{x}_f/\partial \boldsymbol{\lambda}_0^a$ and $\partial \boldsymbol{\lambda}_f/\partial \boldsymbol{\lambda}_0^a$ depend on the elements of the state transition matrix $\boldsymbol{\Omega}(t_f, t_0)$ at final time, governing the state/costate closed-loop system. We first construct the classical state transition matrix for the closed-loop system and subsequently derive a projection matrix to directly determine the sensitivity $d\boldsymbol{\Theta}/d\boldsymbol{\lambda}_0^a$. The closed-loop system in this case is obtained by augmenting the state space with the costate equation:

$$\mathbf{z}\,(t) := \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\lambda}(t) \end{bmatrix}$$

and observing that the control law $H_{\mathbf{u}} = 0$ in most cases, establishes $\mathbf{u} = \mathbf{u}(\boldsymbol{\lambda})$. In this discussion, we concentrate on problems in which the optimal control necessary conditions render control input as an analytic function of the costate equation. Thus, the Eqs. (4) can be written as

$$\dot{\mathbf{z}} = \begin{bmatrix} H_{\boldsymbol{\lambda}} \\ -H_{\mathbf{x}} \end{bmatrix} = \mathbf{g}(\mathbf{z}) \tag{11}$$

with initial conditions

$$\mathbf{z}_0 = \mathbf{z}(t_0) = \begin{bmatrix} \mathbf{x}(t_0) \\ \boldsymbol{\lambda}(t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \boldsymbol{\lambda}_0 \end{bmatrix}$$

The differential equations governing the evolution of this first-order state transition matrix about the trajectory corresponding to the current initial guess [written as $\mathbf{z}(t, \boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0^*)$] are given by

$$\frac{d}{dt}\boldsymbol{\Omega}(t, t_0) = \left[\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right]_{\mathbf{z}(t, \boldsymbol{\lambda}_0 = \boldsymbol{\lambda}^*)} \boldsymbol{\Omega}(t, t_0) \tag{12}$$

with the definition

$$\boldsymbol{\Omega}\,(t, t_0) := \begin{bmatrix} \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_0} & \frac{\partial \mathbf{x}(t)}{\partial \boldsymbol{\lambda}_0} \\ \frac{\partial \boldsymbol{\lambda}(t)}{\partial \mathbf{x}_0} & \frac{\partial \boldsymbol{\lambda}(t)}{\partial \boldsymbol{\lambda}_0} \end{bmatrix} \tag{13}$$

and initial conditions $\boldsymbol{\Omega}(t_0, t_0) = \mathbf{I}_n$, the $n \times n$ identity matrix. To construct a general element in the admissible subspace, we consider the first variation of the of the transversality conditions at initial time t_0 [Eq. (5)] written as

$$\frac{\partial}{\partial \mathbf{x}_0}\left(\frac{\partial \Phi}{\partial x_{j_2}(t_0)}\right)\delta\mathbf{x}_0 + \delta\lambda_{j_2}(t_0) = 0 \tag{14}$$

for $j_2 = q_0 + 1, \ldots, n$. This leads to a linear system of equations

$$[C \quad \mathbf{I}_{n-q_0}]\begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\boldsymbol{\lambda}_{q_0+1:n}(t_0) \end{bmatrix} = [0]_{(n-q_0)\times 1} \tag{15}$$

with the definition

$$C_{i,j} = \left(\frac{\partial^2 \Phi}{\partial x_i(t_0)\partial x_j(t_0)}\right) \in \mathbb{R}^{n-q_0\times n} \tag{16}$$

$\forall\ i = q_0 + 1, \ldots, n$ and $\forall\ j = 1, \ldots, n$. Consequently, there exists the nonzero sensitivity

$$\frac{\partial \mathbf{x}_0}{\partial \boldsymbol{\lambda}_0^a} = \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \boldsymbol{\lambda}_{1:q_0}(t_0)} & \frac{\partial \mathbf{x}_0}{\partial \boldsymbol{\lambda}_{q_0+1:n}(t_0)} \end{bmatrix} = [\,[0]_{n\times q_0} \quad -C^T(CC^T)^{-1}\,] \tag{17}$$

The superscript $\partial\boldsymbol{\lambda}_0^a$ in the preceding sensitivity is written to indicate that the admissibility has been accounted for in the calculation by inverting the condition in Eq. (14) as a result of application of Lagrange implicit function theorem on the tangent plane to the transversality conditions of Eq. (5). Therefore, the correction matrix to make the search span only the admissible subspace is given by

$$\frac{\partial \mathbf{z}_0}{\partial \boldsymbol{\lambda}_0^a} = \begin{bmatrix} \frac{\partial \mathbf{x}_0}{\partial \boldsymbol{\lambda}_0^a} \\ \frac{\partial \boldsymbol{\lambda}_0}{\partial \boldsymbol{\lambda}_0^a} \end{bmatrix} = \begin{bmatrix} [0]_{n\times q_0} & -C^T(CC^T)^{-1} \\ & \mathbf{I}_n \end{bmatrix} \tag{18}$$

where the fact that $\partial\boldsymbol{\lambda}_0/\partial\boldsymbol{\lambda}_0^a = \mathbf{I}_n$ has been employed. Note that the last $n - q_0$ columns of $\partial\mathbf{z}_0/\partial\boldsymbol{\lambda}_0^a$ (i.e., $q_0 + 1, \ldots, n$) form a basis for the range of all $\delta\mathbf{z}_0$ such that

$$[C \quad [0]_{(n-q_0)\times(q_0)} \quad \mathbf{I}_{n-q_0}]\delta\mathbf{z}_0 = [0]_{(n-q_0)\times 1} \tag{19}$$

such that the Newton's method search directions for $\delta\boldsymbol{\lambda}_{q_0+1:n}(t_0)$ are consistent with the transversality condition variations of Eq. (15) (Fredholm alternative theorem [10]). This is a consequence of our choice of the variables for optimization (guess variables). The analysis presented herein hinges greatly on this choice, because for a different choice of optimization variables (say, the unspecified states are being iterated upon), one obtains a different set of equivalent corrections to account for the dependence

$$\frac{d\mathbf{\Theta}}{d\boldsymbol{\lambda}_0^a}=\frac{\partial\mathbf{\Theta}}{\partial\boldsymbol{\lambda}_0^a}+\frac{\partial\mathbf{\Theta}}{\partial\mathbf{z}_f}\left(\frac{\partial\mathbf{z}_f}{\partial\mathbf{z}_0}\right)\frac{\partial\mathbf{z}_0}{\partial\boldsymbol{\lambda}_0^a}=\frac{\partial\mathbf{\Theta}}{\partial\boldsymbol{\lambda}_0^a}+\frac{\partial\mathbf{\Theta}}{\partial\mathbf{z}_f}\mathbf{\Omega}(t_f,t_0)\frac{\partial\mathbf{z}_0}{\partial\boldsymbol{\lambda}_0^a} \tag{20}$$

with the definition of the correction matrix $\partial\mathbf{z}_0/\partial\boldsymbol{\lambda}_0^a$ as in Eq. (18). Let us now demonstrate the method on an orbit transfer problem.

Example Application: Orbit Transfer Problem

Consider the interplanetary orbit transfer problem in which it is required to reach an asteroid orbit from Earth in a given amount of transfer time. The target orbit was obtained by rotating the orbit plane of the asteroid Apophis [11] to be coplanar with the ecliptic (for simplistic illustration in the current discussion). The earliest launch time frame of interest was assumed to be in January 2011. To gain initial insight into the physics of the problem and to ascertain convergence uniformly with a single initial guess on the initial costate, we report a sample solution from a larger solution set obtained by considering a launch for each of 150 consecutive days, starting 1 January 2011, and times of flight to Apophis ranging from 130 through 200 days. The entire solution set and the associated parameter sensitivity calculations have been reported in [6]. The transfer maneuver is considered to be a low-thrust transfer, but with allowable impulses at the initial and final times. Consequently, we seek a solution that minimizes the impulse requirements at those times and also thrust to steer the spacecraft matching the fixed boundary conditions. The problem considered can be formally stated as

$$\begin{aligned}\min\tfrac{1}{2}((u(t_f)-u_{\text{asteroid}})^2+(\upsilon(t_f)-\upsilon_{\text{asteroid}})^2)\\ +\tfrac{1}{2}((u(t_0)-u_{\text{Earth}})^2+(\upsilon(t_0)-\upsilon_{\text{Earth}})^2)\end{aligned} \tag{21}$$

subject to

$$\dot r=u,\qquad \dot\theta=\frac{\upsilon}{r},\qquad \dot u=\frac{\upsilon^2}{r}-\frac{\mu}{r^2}+\frac{T\sin(\phi)}{m_0-|\dot m|t}$$
$$\dot\upsilon=-\frac{u\upsilon}{r}+\frac{T\cos(\phi)}{m_0-|\dot m|t} \tag{22}$$

with terminal boundary conditions

$$\Psi_{f1}=\cos(\theta(t_f)-\theta_{\text{asteroid at }t_f})-1=0$$
$$\Psi_{f2}=r(t_f)-r_{\text{asteroid at }t_f}=0 \tag{23}$$

where T is the constant thrust parameter, $\phi(t)$ is the flight-path-angle time history, m_0 is the initial mass of the spacecraft, $|\dot m|$ is the mass flow rate from the thruster of the spacecraft, $r(t)$ is the radius magnitude, $\theta(t)$ is the true anomaly, $u(t)$ is the radial component of the velocity, and $\upsilon(t)$ is the transverse component.

The initial conditions are specified as

$$r(t_0)=r_{\text{Earth at }t_0},\qquad \theta(t_0)=\theta_{\text{Earth at }t_0} \tag{24}$$

with specified initial and terminal times t_0 and t_f. The Hamiltonian for the problem is

$$\begin{aligned}H=\lambda_r u+\lambda_\theta\frac{\upsilon}{r}+\lambda_u\left[\frac{\upsilon^2}{r}-\frac{\mu}{r^2}+\frac{T\sin\phi}{m_0-|\dot m|t}\right]\\ +\lambda_\upsilon\left[-\frac{u\upsilon}{r}+\frac{T\cos\phi}{m_0-|\dot m|t}\right]\end{aligned} \tag{25}$$

leading to the differential equations for the costate variables being given by

$$\dot\lambda_r=-\lambda_u\left(-\frac{\upsilon^2}{r^2}+\frac{2\mu}{r^3}\right)-\lambda_\upsilon\left(\frac{u\upsilon}{r^2}\right)+\lambda_\theta\frac{\upsilon}{r^2},\qquad \dot\lambda_\theta=0$$
$$\dot\lambda_u=-\lambda_r+\lambda_\upsilon\frac{\upsilon}{r},\qquad \dot\lambda_\upsilon=-\lambda_u\frac{2\upsilon}{r}+\lambda_\upsilon\frac{u}{r}-\frac{\lambda_\theta}{r} \tag{26}$$

The optimal flight-path angle is determined from the necessary condition

$$H_\phi=\lambda_u\frac{T\cos\phi}{m_0-|\dot m|t}-\lambda_\upsilon\frac{T\sin\phi}{m_0-|\dot m|t}=0 \tag{27}$$

Note that for this problem that $q_f=q_0=2$ and that the transversality conditions at initial and terminal times specify the velocity mismatch at the asteroid (t_f) and the minimum burnout mismatch at the launch (t_0). Canonical units have been employed in the calculations [12] such that the gravitational parameter $\mu=1$.

The optimal steering angle, following Eq. (27), is given by $\phi_{\text{extremal}}=\tan^{-1}(\lambda_u/\lambda_\upsilon)$, and the transversality conditions corresponding to the unknowns states are given by

$$\begin{aligned}\lambda_u(t_0)&=-\frac{\partial\phi}{\partial u_0}=-(u_0-u_{\text{Earth}})\\ \lambda_\upsilon(t_0)&=-\frac{\partial\phi}{\partial\upsilon_0}=-(\upsilon_0-\upsilon_{\text{Earth}})\\ \lambda_u(t_f)&=\frac{\partial\phi}{\partial u_f}=(u_f-u_{\text{asteroid}})\\ \lambda_\upsilon(t_f)&=\frac{\partial\phi}{\partial\upsilon_0}=(\upsilon_f-\upsilon_{\text{asteroid}})\end{aligned} \tag{28}$$

We audit the conditions (transversality and given boundary conditions) and unknowns (initial costates) as follows. At t_f [the equation corresponding to Eq. (7) for this problem],

$$\mathbf{\Theta}:=\begin{bmatrix}\cos(\theta(t_f)-\theta_{\text{asteroid at }t_f})-1\\ r(t_f)-r_{\text{asteroid at }t_f}\\ \lambda_u(t_f)-(u_f-u_{\text{asteroid}})\\ \lambda_\upsilon(t_f)-(\upsilon_f-\upsilon_{\text{asteroid}})\end{bmatrix} \tag{29}$$

Although the explicit partials and the state costate closed-loop system can be written taking all the required partials, the required correction matrix is given by [equivalent of Eq. (18) for the example problem]

$$\frac{\partial\mathbf{z}_0}{\partial\boldsymbol{\lambda}_0^a}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \tag{30}$$

where

$$C=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}$$

for the present problem.

Notice the nonlinear hard boundary constraint associated with the true anomaly of terminal time in Eqs. (29) and (23). The specification presented herein naturally tackles the phase problem, because the true anomaly specification is typically phased from the propagated solution in multiples of 2π, depending on the conventions involved.

Table 1 Parameters used for numerical simulation (cf. [14,15])

Parameter name	Value
Initial time t_0	35th day following 1 Jan. 2011
Time of flight Δt	175 days
Thrust T	3.784 N
Initial mass m_0	4545.5 kg
Fuel-consumption rate $\lvert\dot m\rvert$	6.7744×10^{-5} kg/s

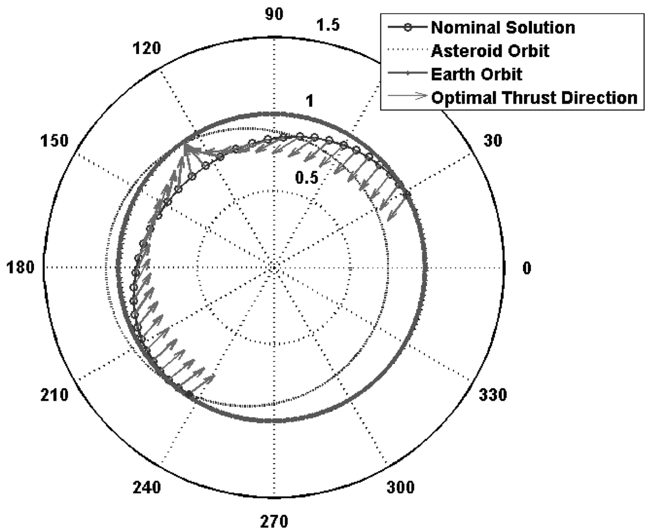

Fig. 1 Solution of TPBVP: orbit transfer example.

A numerical solution to this example application problem was obtained using the method developed in this paper. Parameters for the numerical simulation are outlined in Table 1. Terminal constraint satisfaction error (for a specified tolerance of 1e − 7) and the converged initial costate are given by

$$\boldsymbol{\psi}_f = \begin{bmatrix} -1.4439\mathrm{e}-009 \\ -1.2926\mathrm{e}-008 \\ -2.1845\mathrm{e}-009 \\ -2.5846\mathrm{e}-008 \end{bmatrix}, \qquad \boldsymbol{\lambda}_0^{\text{converged}} = \begin{bmatrix} 0.11637 \\ -0.029603 \\ 0.078305 \\ 0.035679 \end{bmatrix} \tag{31}$$

The trajectory obtained by solving the TPBVP is plotted in Fig. 1.

The initial guess used for the costate vector is given by

$$\boldsymbol{\lambda}_0 = [0.0462 \quad -0.0089 \quad 0.0278 \quad 0.0268]^T \tag{32}$$

Alternatively, a naïve implementation of the Newton's method without the correction factor developed in this paper was found to diverge (with the same initial guesses for the costate vector). Standard nonlinear solvers (a MATLAB version of MINPACK [13]) obtain the same solution as provided by the modification presented herein. It is known that the nonlinear equation solvers construct the Jacobian via function evaluations and finite differences. Thus, the reported results demonstrate the utility of the correction developed in the paper to the classical shooting method, making use of the Lagrange implicit function theorem.

Conclusions

The Lagrange implicit function theorem is applied to develop a numerical iteration (of Newton type) procedure for the solution of two-point boundary-value problems in optimal control. Further, a correction matrix is derived to make the corrections of Newton's method remain within an admissible subspace of the known transversality manifold available from the necessary conditions of the optimal control problem. The procedure thus set up is applied to an orbit transfer problem. The modification to Newton iterates derived from the implicit function theorem therefore provides us optimism about the utility of the theory of implicit functions and its generalizations in similar problems of dynamic optimization.

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